Gravitons from a loop representation of linearised gravity

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ABSTRACT

Loop quantum gravity is based on a classical formulation of 3+1 gravity in terms of a real $SU(2)$ connection. Linearization of this classical formulation about a flat background yields a description of linearised gravity in terms of a *real* $U(1) \times U(1) \times U(1)$ connection. A 'loop' representation, in which holonomies of this connection are unitary operators, can be constructed. These holonomies are not well defined operators in the standard graviton Fock representation. We generalise our recent work on photons and $U(1)$ holonomies to show that Fock space gravitons are associated with distributional states in the $U(1) \times U(1) \times U(1)$ loop representation. Our results may illuminate certain aspects of the much deeper (and as yet unkown,) relation between gravitons and states in nonperturbative loop quantum gravity.
This work leans heavily on earlier seminal work by Ashtekar, Rovelli and Smolin (ARS) on the loop representation of linearised gravity using *complex* connections. In the last part of this work, we show that the loop representation based on the *real* $U(1) \times U(1) \times U(1)$ connection also provides a useful kinematic arena in which it is possible to express the ARS complex connection- based results in the mathematically precise language currently used in the field.

1. Introduction

Loop quantum gravity [1, 2, 3] is an attempt to apply standard Dirac quantization techniques to a classical Hamiltonian formulation of 3+1 gravity in which the basic variables are a spatial $SU(2)$ connection and its conjugate triad field. In addition to the usual diffeomorphism and Hamiltonian constraints, the formulation also has an $SU(2)$ Gauss law constraint which ensures that triad rotations are gauge. At the $SU(2)$ gauge invariant level (also referred to as the kinematic level), the representation space is generated by the action of (traces of) holonomies of the connection on a cyclic state. Since holonomies are labelled by 1 dimensional, arbitrarily complicated loops, the basic quantum excitations may be visualised as 1 dimensional and 'polymer- like'. Physical states, which are in the kernel of all the constraints, are expressible as certain kinematically non- normalizable, linear combinations of these polymer-like excitations [4].

A key open question is: how do classical configurations of the gravitational field arise ? In particular, how does flat spacetime (and small perturbations around it) arise from non-perturbative quantum states of the gravitational field? The latter question is particularly interesting for the following reason. Small perturbations about flat spacetime correspond to solutions of linearized gravity. Quantum states of linearised gravity lie in the familiar graviton Fock space on which the conventional perturbative approaches to quantum gravity are based. Such approaches seem to fail due to nonrenormalizability problems. Thus, an understanding of the relation between the quantum states of linearised gravity and states in full nonperturbative loop quantum gravity would shed light on the reasons behind the failure of perturbative methods.

In this work we focus exclusively on understanding certain structures in quantum linearised gravity, which concievably (but by no means, assuredly!) could play a role in the much deeper issue of the relation between perturbative and non-perturbative states. Our starting point is the linearization of the classical $SU(2)$ formulation [5] on which loop quantum gravity is based. This linearization is described in section 2 wherein we also show that the linearized Gauss Law constraint generates $U(1) \times$

$U(1)\times U(1)$ transformations on the linearized connection.

Using the methods of [6, 7, 8, 9, 10], $U(1)^3$ counterparts of the $SU(2)$ based structures of loop quantum gravity can be constructed. In particular, at the $U(1)^3$ gauge invariant level, a 'kinematic' Hilbert space $\mathcal{H}_{kin}$ exists which is spanned by 1 dimensional polymer-like excitations associated with (triplets [11] of) loops. Holonomies of the linearised connection are represented as unitary operators on $\mathcal{H}_{kin}$. We exhibit this representation in section 3a.

As realised in [11], the operator corresponding to the magnetic field of the linearised connection plays a key role in expressing the linearised diffeomorphism and Hamiltonian constraints as quantum operators. It turns out that this operator is not well defined in $\mathcal{H}_{kin}$. Neverthless it can be represented on a vector space Φ^{*L}_{kin} of appropriately well behaved distributional combinations of elements in $\mathcal{H}_{kin}$. Using this representation of the magnetic field operator, we identify the kernel, Φ^{*L}_{phys}, of all the constraints. Since $\Phi^{*L}_{phys}\subset\Phi^{*L}_{kin}$, elements of Φ^{*L}_{phys} are also associated with infinite, kinematically non-normalizable sums of 1 dimensional polymer-like excitations. Section 3b is devoted to a discussion of the magnetic field operator and an evaluation of the kernel of the quantum constraints.

The standard graviton Fock space representation of linearised gravity is very different from the above 'loop' representation. The basic excitations in Fock space are 3d and wavelike in contrast to the polymer-like nature of excitations in the loop representation. Moreover, in the Fock representation the connection is an operator valued distribution which needs to be smeared in 3 dimensions to obtain a well defined operator. Since holonomies involve only 1 dimensional smearings along loops, they are not well defined operators on Fock space.

In view of the above remarks, it is a non-trivial task to relate Fock space gravitons to elements in Φ^{*L}_{phys}. In section 4, we generalise the considerations of [10] to relate the loop representation of linearised gravity to its standard Fock representation. As in [10] we use the Poincare invariance of the Fock vacuum to identify graviton states in Φ^{*L}_{phys}. [1]

[1]In this paragraph (but *not* in section 4) we gloss over the very important difference between

Recall that the starting point of this work is the linearization of a *real* $SU(2)$ formulation [5, 12] of classical gravity. The basic variable is a real $SU(2)$ connection and the associated Barbero-Immirzi parameter [5, 13] is real. In contrast Ashtekar, Rovelli and Smolin use the complex self dual Ashtekar-Sen connection [14] in their pioneering work [11] on a loop representation of linearised gravity. This corresponds to the choice of an imaginary Barbero-Immirzi parameter. In section 5, we show how to extend the considerations of sections 2- 4 to the case of an arbitrary complex Barbero- Immirzi parameter. Section 6 is devoted to a discussion of our results.

As mentioned above, our real interest is in the deeper issue of the relation between states in linearised gravity and in full quantum gravity rather than just in structures in linearised gravity. One possible way to approach the deeper issue is to divide it into two parts. First, since both loop quantum gravity and the $U(1)^3$ representation are structurally similar, we may try to relate the two. This is the really hard part. The second (and much easier) part is to relate the $U(1)^3$ representation to the standard graviton Fock representation. It is only the second part that we accomplish in this paper.

This work is heavily based on the Ashtekar-Rovelli-Smolin paper [11] and on [10]. For this reason, we shall be very brief in our presentation and sketch only the important points. The reader may consult [11, 10] for more details. Indeed, this work may be read as a mathematically precise formulation of the earlier ARS [11] work in the context of the subsequent developments in the field as reflected in, for example, [4, 12, 7, 6, 15, 16, 17, 18, 19, 10, 20].

We use units in which Newton's constant, Planck's constant and the speed of light are unity.

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the standard Fock representation and the r- Fock representation [10] for reasons of brevity and pedagogy.

2. Classical linearised gravity as theory of $U(1) \times U(1) \times U(1)$ connections.

Our starting point is the Hamiltonian formulation of 3+1 gravity discussed in [5] The spacetime manifold has topology $\Sigma \times R$ where Σ is a 3 dimensional orientable manifold. The phase space variables are a spatial $SU(2)$ connection, $A_a^i(\vec{x})$ and a densitized triad field $E_j^b(\vec{y})$. Here a, b denote spatial components, i, j denote internal $SU(2)$ Lie algebra components and $\vec{x}, \vec{y}$ denote (in general, local) coordinates on Σ. The only non- vanishing Poisson bracket is

$$\{A_a^i(\vec{x}), E_j^b(\vec{y})\} = \frac{\gamma_0}{2}\delta_a^b\delta_j^i\delta(\vec{x}, \vec{y}). \tag{1}$$

Here, γ_0 is the Barbero-Immirzi parameter [5, 13]. The spin connection associated with the triad field is denoted by Γ_a^i, the curvature of A_a^i by F_{ab}^i and the gauge covariant derivative associated with A_a^i by $\mathcal{D}_a$. The constraints of the theory are the Gauss law constraint G_i, the vector or diffeomorphism constraint V_a and the Hamiltonian constraint C. They are given by

$$\begin{aligned} G_i &= \mathcal{D}_a E_i^a, & (2)\\ V_a &= E_i^a F_{ab}^i, & (3)\\ C &= \epsilon^{ijk} E_i^a E_j^b F_{abk} - 2\frac{\gamma_0^2+1}{\gamma_0^2} E_{[i}^a E_{j]}^b (A_a^i - \Gamma_a^i)(A_a^j - \Gamma_a^j). & (4) \end{aligned}$$

The $SU(2)$ variables are related to the ADM variables as follows. The densitized triad and the 3- metric, q_{ab}, are related through

$$qq^{ab} = E^{ai}E_i^b \tag{5}$$

where q is the determinant of q_{ab}. When $G_i = 0$ the extrinsic curvature, K_{ab}, can be extracted from the $SU(2)$ variables through

$$\gamma_0 K_{ab} E_i^b = \sqrt{q}(A_a^i - \Gamma_a^i). \tag{6}$$

To define the linearised theory about a flat background we choose $\Sigma = R^3$ and fix, once and for all, a cartesian coordinate system $\{\vec{x}\}$ as well as an orthonormal

basis in the Lie algebra of $SU(2)$. Henceforth all components refer to this cartesian coordinate system and to this internal basis. We linearise the $SU(2)$ formulation about the phase space point ($A_a^i = 0, E_i^a = \delta_i^a$). As in [11], we denote the fluctuation in the triad field by e_i^a so that

$$E_i^a = \delta_i^a + e_i^a. \tag{7}$$

Since the background connection vanishes, there is no need to introduce a new symbol for the fluctuation in the connection. The Poisson brackets between the linearised variables are induced from (1). The only non-vanishing Poisson bracket is

$$\{A_a^i(\vec{x}), e_j^b(\vec{y})\}_L = \frac{\gamma_0}{2}\delta_a^b \delta_j^i \delta(\vec{x}, \vec{y}). \tag{8}$$

Here the subscript 'L' denotes the fact that the Poisson bracket is for linearised theory.

Note that the flat spatial metric corresponding to the background triad is just the Kronecker delta, δ_{ab}. In what follows spatial indices are lowered and raised with this flat metric and its inverse. Internal indices are, of course, lowered and raised by the $SU(2)$ Cartan-Killing metric. We also use the background triad to freely interchange internal and spatial indices. The flat derivative operator which annihilates the background triad is denoted by ∂_a.

The linearised constraints are obtained from (2), (3)and (4) by keeping terms at most linear in the fluctuations and are denoted by G_i^L, V_a^L, and C^L with

$$G_i^L = \partial_a e_i^a + \epsilon_i^{\ ja} A_{aj}, \tag{9}$$

$$V_a^L = f_{ab}^a, \tag{10}$$

$$C^L = \epsilon^{abc} f_{abc}. \tag{11}$$

Here $f_{ab}^i = \partial_a A_b^i - \partial_b A_a^i$ is the linearised curvature.

The transformations generated by $G^L(\Lambda) := \int d^3x \Lambda^i G_i^L$ are

$$\delta A_a^i = \{A_a^i, G^L(\Lambda)\} = -\partial_a (\frac{\gamma_0 \Lambda^i}{2}) \tag{12}$$

and

$$\delta e_i^a = \{e_i^a, G^L(\Lambda)\} = -\epsilon_i^{\ ak} (\frac{\gamma_0 \Lambda_k}{2}). \tag{13}$$

From (12), A_a^i for each 'i' transforms as a $U(1)$ connection. Thus, the configuration space of linearised gravity in this formulation is coordinatized by a triplet of $U(1)$ connections A_a^1, A_a^2, A_a^3.

In order to construct the loop representation in the next section, we define the following set of G_i^L- invariant functions on phase space:

$$h^{ab} = e^{ab} + e^{ba}, \tag{14}$$

$$H_\alpha^k = \exp i \oint_\alpha A_a^k dx^a. \tag{15}$$

Here α is a piecewise analytic, oriented loop in R^3 and H_α^k is the $U(1)$ holonomy of A_a^k around the loop α.

It is also useful to define the magnetic field of the connection by

$$B_k^a = \frac{1}{2}\epsilon^{abc} f_{bck}. \tag{16}$$

In terms of the magnetic field the vector and scalar constraints are

$$V_a^L = \epsilon_{cab} B^{ca} \tag{17}$$

$$C^L = B_c^c. \tag{18}$$

Thus the vanishing of the vector and scalar constraints imply that the magnetic field is symmetric and tracefree.

3. The 'loop' representation of quantum linearised gravity

We construct a loop representation based on the $U(1)^3$ holonomies of section 2. The representation at the kinematic (G_i^L invariant) level is just the tensor product of 3 copies of the $U(1)$ representation worked out in detail in [10]. We use the notation of, and assume familiarity with that work.

After presenting the kinematic Hilbert space in section 3a, we turn our attention to the linearised vector and scalar constraints in section 3b. Since the constraints are algebraic statements about the magnetic field, we express the classical magnetic

field via a limit of the holonomy of a shrinking loop in the usual way. The corresponding quantum operator is not defined on the kinematic Hilbert space because the diffeomorphism invariance of the Hilbert space measure precludes the existence of the required limit. We show how to define the magnetic field operator based on the dual action of the holonomy operator on a suitable space of distributions. We use this defintion to find the kernel of the linearised vector and scalar constraints.

3a. The kinematic Hilbert space representation

The kinematic Hilbert space, $\mathcal{H}_{kin}$, inherits its measure from the Haar measure on $U(1)$ (it is just the triple product of the Ashtekar-Lewandowski measure for $U(1)$ connections [7]). A spanning orthonormal basis is given by the triple tensor product of the $U(1)$ flux network basis of [10]. [2] Each basis state is labelled by a triplet of closed, oriented, piecwise analytic graphs as well as 3 sets of integers (these are representation labels for $U(1)$), one for each graph of the triplet. Each set of integers labels the edges of its corresponding graph in such a way that at every vertex the sum of labels of outgoing edges equals the sum of labels of incoming edges.

We denote the flux network labelled by the graphs α_i and the sets of integers q_i, $i = 1..3$ as

$$|\boldsymbol{\alpha}, \{\mathbf{q}\} >= |\alpha_1\{q_1\} > |\alpha_2\{q_2\} > |\alpha_3\{q_3\} > . \tag{19}$$

As shown in [10], the $U(1)$ holonomy of any piecewise analytic loop β is equally well associated with a $U(1)$ flux network label $\alpha, \{q\}$ such that

$$X^a_{\alpha,\{q\}}(\vec{x}) = X^a_\beta(\vec{x}). \tag{20}$$

Here

$$X^a_\beta(\vec{x}) := \oint_\beta ds \delta^3(\vec{\beta}(s), \vec{x})\dot{\beta}^a, \tag{21}$$

and

$$X^a_{\alpha,\{q\}}(\vec{x}) := \sum_{I=1}^{N} q_I \int_{e_I} ds_I \delta^3(\vec{e}_I(s_I), \vec{x})\dot{e_I}^a. \tag{22}$$

[2]In [10] this was called the charge network basis; we use the term flux network to agree with the more recent work [22].

where e_I is the Ith edge of α and is labelled by the integer q_I.

The gauge invariant operators $\hat{H}^i$ and $\hat{h}^{ab}$ are represented on the kinematic Hilbert space as follows. We first describe the action of $\hat{H}^1$. This operator acts only on the first ket on the right hand side of (19) exactly as in the case of $U(1)$ theory [10]. Recall, from [10] that, there, the $U(1)$ operator $\hat{H}_{\eta,\{p\}}$ maps $|\alpha, \{q\}>$ to a new flux network state based on the graph $\alpha \cup \eta$ consisting of the union of the sets of edges belonging to α and η. [3] The edges of $\alpha \cup \eta$ are oriented and labelled with integers as follows. Edges which are not shared by η and α retain their orientations and labels. Any shared edge labelled by the integer q in α retains its orientation from α and has the label $q+p$ if it has the same orientation in η and the label $q-p$ if it has opposite orientation in η. The new state is denoted (with a minor change of notation with respect to [10]) by $|\alpha, \{q\} \cup \eta, \{p\}>$. This implies that in the $U(1)^3$ case we have,

$$\hat{H}^1_{\eta,\{p\}}|\boldsymbol{\alpha}, \{\mathbf{q}\}> = |\alpha_1, \{q_1\} \cup \eta, \{p\}> |\alpha_2, \{q_2\}> |\alpha_3, \{q_3\}> . \tag{23}$$

Similarly $\hat{H}^2, \hat{H}^3$ act by the union operation on the labels $\alpha_2\{q_2\}$ and $\alpha_3\{q_3\}$. Using the notation of [11], we write

$$\hat{H}^k_{\eta,\{p\}}|\boldsymbol{\alpha}, \{\mathbf{q}\}> = |\boldsymbol{\alpha}, \{\mathbf{q}\} \cup_k \eta, \{p\}> . \tag{24}$$

As in [10] we shall use the labelling of holonomies by their associated flux network labels (i.e. $H^k_{\alpha\{q\}}$) interchangeably with their labelling by loops (i.e. H^k_β). Thus if there is no integer label in the subscript to H, the label is to be understood as a loop else as an associated flux network label.

Also, note that if β is a loop with a single edge, then the associated flux network label comprises of the graph β with its single edge labelled by the integer 1. For this special case we write

$$H^k_\beta = H^k_{\beta,\{1\}}. \tag{25}$$

$\hat{h}^{ab}$ is represented as

$$\hat{h}^{ab}(\vec{x})|\boldsymbol{\alpha}, \{\mathbf{q}\}> = \gamma_0 X^{(ab)}_{\boldsymbol{\alpha},\{\mathbf{q}\}}(\vec{x}) \tag{26}$$

[3]It is assumed that edges of η, α overlap only if they are identical and that intersections of η, α occur only at vertices of η, α . This entails no loss of generality, since we can always find graphs which are holonomically equivalent to η, α and for which the assumption holds.

where we have defined

$$X^{ab}_{\boldsymbol{\alpha},\{\mathbf{q}\}}(\vec{x}) = \sum_{i=1}^{3} X^{a}_{\alpha_i,\{q_i\}}(\vec{x})\delta^{b}_{i} \tag{27}$$

It can be verified that (24) and (26) provide a * representation (on the kinematic Hilbert space) of the Poisson bracket algebra of the G^L_i invariant functions H^k_α and $h_{ab}(\vec{x})$. Therefore the linearised Gauss law constraint is already taken care of and we need to analyse only the remaining (quantum) vector and scalar constraints.

3b. The Magnetic field operator and physical states

The magnetic field is extracted from the holonomies of small loops through

$$B^{ck}(\vec{x}) = \lim_{\delta\to 0} \frac{i}{\pi\delta^2}(H^k_{(\gamma^c_{\vec{x},\delta})^{-1}} - 1) \tag{28}$$

where $*$ denotes complex conjugation and $\gamma^c_{\vec{x},\delta}$ is a circular loop of radius δ centered at $\vec{x}$ traversing anticlockwise about and with its plane normal to, the 'c' axis. $(\gamma^c_{\vec{x},\delta})^{-1}$ denotes the same loop running clockwise. The corresponding operator

$$\hat{B}^{ck}(\vec{x}) = \lim_{\delta\to 0} \frac{i}{\pi\delta^2}(\hat{H}^k_{(\gamma^c_{\vec{x},\delta})^{-1}} - 1) \tag{29}$$

is not well defined on the finite span of flux network states. The reason is that, due to the diffeomorphism invariance of the $U(1)^3$ Ashtekar-Lewandowski measure, flux network states associated with the triplet of graphs $\boldsymbol{\alpha} \cup_k (\gamma^c_{\vec{x},\delta})^{-1}$ (here we use the notation of [11]) for different values of δ are orthogonal.

Instead we attempt to define the operator $\hat{B}^{ck}$ by its dual action on the space of algebraic duals to the finite span of flux network states. Recall that the dual (anti-)representation of an operator $\hat{A}$ is given by [10]

$$\hat{A}\Phi(|\psi>) = \Phi(\hat{A}^\dagger|\psi>) \tag{30}$$

where Φ is an element of the algebraic dual and $|\psi>$ is a finite linear combination of flux network states. Every element of the algebraic dual can be formally written as an infinite sum over all flux network states i.e.

$$\Phi := \sum_{\boldsymbol{\alpha},\{\mathbf{q}\}} c_{\boldsymbol{\alpha},\{\mathbf{q}\}} < \boldsymbol{\alpha},\{\mathbf{q}\}| \tag{31}$$

with $c_{\boldsymbol{\alpha},\{\mathbf{q}\}} = \Phi(|\boldsymbol{\alpha}, \{\mathbf{q}\} >)$. It follows that

$$\hat{B}^{ck}(\vec{x})\Phi = \lim_{\delta\to 0}\sum_{\boldsymbol{\alpha},\{\mathbf{q}\}} c_{\boldsymbol{\alpha},\{\mathbf{q}\}} < \boldsymbol{\alpha}, \{\mathbf{q}\}| \frac{(\hat{H}^{\dagger}_{(\gamma^c_{\vec{x},\delta})^{-1}} - 1)}{i\pi\delta^2}$$

$$= \sum_{\boldsymbol{\alpha},\{\mathbf{q}\}} \lim_{\delta\to 0} \frac{c_{\boldsymbol{\alpha},\{\mathbf{q}\}\cup_k \gamma^c_{\vec{x},\delta},\{1\}} - c_{\boldsymbol{\alpha},\{\mathbf{q}\}}}{i\pi\delta^2} < \boldsymbol{\alpha}, \{\mathbf{q}\}|. \tag{32}$$

We shall say that $\hat{B}^{ck}(\vec{x})$ is well defined iff

$$\frac{\delta c_{\boldsymbol{\alpha},\{\mathbf{q}\}}}{\delta\gamma^{ck}(\vec{x})} := \lim_{\delta\to 0} \frac{c_{\boldsymbol{\alpha},\{\mathbf{q}\}\cup_k \gamma^c_{\vec{x},\delta},\{1\}} - c_{\boldsymbol{\alpha},\{\mathbf{q}\}}}{\pi\delta^2} \tag{33}$$

is well defined.

We further require that $c_{\boldsymbol{\alpha},\{\mathbf{q}\}}$ is a functional of $X^a_{\alpha_i,\{q_i\}}, i = 1..3$. This requirement combined with the requirement that $\hat{B}^{ck}$ be well defined singles out a vector space, Φ^{*L}_{kin}, of 'well behaved' distributions. To summarise: the magnetic field operator is defineable, via the dual action of holonomy operators, on the space Φ^{*L}_{kin}.

The Fourier transform of $X^b_{\alpha_i,\{q_i\}}(\vec{x})$ is

$$X^b_{\alpha_i,\{q_i\}}(\vec{p}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3x X^b_{\alpha_i,\{q_i\}}(\vec{x}) e^{-i\vec{p}\cdot\vec{x}}. \tag{34}$$

Define

$$\frac{\delta X^b_{\alpha_j,\{q_j\}}(\vec{p})}{\delta\gamma^{ci}(\vec{x})} := \delta_{ij} \lim_{\delta\to 0} \frac{X^b_{\alpha_j,\{q_j\}\cup\gamma^c_{\vec{x},\delta},\{1\}} - X^b_{\alpha_j,\{q_j\}}}{\pi\delta^2}. \tag{35}$$

From (22) it follows that

$$\frac{\delta X^b_{\alpha_j,\{q_j\}}(\vec{p})}{\delta\gamma^{ci}(\vec{x})} = \frac{-i}{(2\pi)^{\frac{3}{2}}} \delta_{ij} p_m \epsilon^{cmb} e^{-i\vec{p}\cdot\vec{x}}. \tag{36}$$

Note that

$$\frac{\delta c_{\boldsymbol{\alpha},\{\mathbf{q}\}}}{\delta\gamma^{ck}(\vec{x})} = \int d^3p \frac{\delta c_{\boldsymbol{\alpha},\{\mathbf{q}\}}}{\delta X^b_{\alpha_j,\{q_j\}}(\vec{p})} \frac{\delta X^b_{\alpha_j,\{q_j\}}(\vec{p})}{\delta\gamma^{ci}(\vec{x})}. \tag{37}$$

Using (36) and (37) in (32) and taking the Fourier transform of $\hat{B}^{ck}(\vec{x})$, we obtain

$$\hat{B}^{ck}(\vec{p}) = \sum_{\boldsymbol{\alpha},\{\mathbf{q}\}} \frac{\delta c_{\boldsymbol{\alpha},\{\mathbf{q}\}}}{\delta X^b_{\alpha_j,\{q_j\}}(-\vec{p})} < \boldsymbol{\alpha}, \{\mathbf{q}\}|. \tag{38}$$

It is straightforward to show that the constraints in the form (17) and (18)imply that $c_{\boldsymbol{\alpha},\{\mathbf{q}\}}$ depends only on the symmetric, transverse, traceless (STT) part of $X^{bc}_{\boldsymbol{\alpha},\{\mathbf{q}\}}$ (the latter is defined in (27)). In the standard helicity basis of transverse vectors $m_a, \bar{m}_a$ ([11]) the STT part of $X^{bc}_{\boldsymbol{\alpha},\{\mathbf{q}\}}$ can be written as

$$X^{ab(STT)}_{\boldsymbol{\alpha},\{\mathbf{q}\}}(\vec{k}) = X^{+}_{\boldsymbol{\alpha},\{\mathbf{q}\}}(\vec{k})m^a m^b + X^{-}_{\boldsymbol{\alpha},\{\mathbf{q}\}}(\vec{k})\bar{m}^a\bar{m}^b. \tag{39}$$

Denote the space of physical states by Φ^{*L}_{phys}. Then we have shown that $\Phi \in \Phi^{*L}_{phys}$ iff the coefficients $c_{\boldsymbol{\alpha},\{\mathbf{q}\}}$ in (31) are functionals only of $X^{+}_{\boldsymbol{\alpha},\{\mathbf{q}\}}(\vec{k})$ and $X^{-}_{\boldsymbol{\alpha},\{\mathbf{q}\}}(\vec{k})$.

4. The relation between gravitons and states in Φ^{*L}_{phys}

The abelian Poisson bracket algebra of holonomies plays a crucial role in the construction of $\mathcal{H}_{kin}$ [20, 10]. As mentioned in section 1, holonomy operators are not well defined on the standard graviton Fock space. However, suitably defined "smeared holonomies" *are* well defined operators on Fock space [20, 11]. It was noticed in [20, 10] that for the $U(1)$ case, the algebra of smeared holonomies is isomorphic to the holonomy algebra. This isomorphism was used to construct a representation, indistinguishable [4] from the Fock representation, in which holonomies are well defined operators. Since holonomy operators are defined in the $U(1)$ loop representation (called the 'qef' representation in [10]) as well as the new 'r- Fock' representation in [10] (here r is a length scale used to define the smearing), it was possible to relate the r-Fock representation to the loop representation in [10]. The considerations of [10] can be extended to the case of linearised gravity in an obvious and straightforward manner and we shall only present the main results of such an extension in this section.

In section 4a, we briefly review the standard graviton Fock space representation based on linearised ADM variables. In section 4b, we use the Poincare invariance condition [10] to identify the element of Φ^{*L}_{phys} which corresponds to the r- Fock vacuum. We expect that this identification can then be used to relate a suitable

[4] Whether (in the $U(1)$ case) indistinguishable even in principle or only practically indistinguishable at scales large compared to the smearing scale, is discussed in [10].

subspace of Φ^{*L}_{phys} to (a dense subspace of) the r- Fock space, modulo a couple of open technical issues which we discuss in section 4c.

4a. Review of the standard Fock space representation of linearised gravity

The standard Fock representation is obtained by quantizing the true degrees of freedom in the ADM description. In the ADM description the phase space variables are the linearised metric , α_{ab} and the linearised ADM momentum, P^{ab} with

$$\{\alpha_{ab}(\vec{x}), P^{cd}(\vec{y})\} = \delta_a^{(c}\delta_b^{d)}\delta(\vec{x},\vec{y}). \tag{40}$$

The true degrees of freedom are parametrised by the transverse, traceless part of α_{ab} and P^{cd} and are denoted by α^{TT}_{ab} and P^{cdTT}. The true Hamiltonian is

$$H_L = \int d^3x(\frac{\partial_m\alpha^{TT}_{cd}}{2}\frac{\partial^m\alpha^{cdTT}}{2} \;\;+\;\; P^{cdTT}P^{TT}_{cd}) \tag{41}$$

so that

$$\dot{\alpha}^{TT}_{cd} = 2P^{TT}_{cd}, \quad \dot{P}^{TT}_{cd} = \frac{\partial^m\partial_m\alpha^{TT}_{cd}}{2}. \tag{42}$$

These evolution equations together imply that

$$\Box\alpha^{TT}_{cd} = 0 \tag{43}$$

which in turn implies that α^{TT}_{cd} has the following plane wave expansion

$$\begin{aligned}\alpha^{TT}_{cd}(\vec{x},t) \;\;=\;\; & \frac{1}{(2\pi)^{\frac{3}{2}}}\int\frac{d^3k}{\sqrt{k}}(a_{(+)}(\vec{k})e^{i(\vec{k}\cdot\vec{x}-kt)}m_cm_d + a^*_{(+)}(\vec{k})e^{-i(\vec{k}\cdot\vec{x}-kt)}\bar{m}_c\bar{m}_d \\ & + a_{(-)}(\vec{k})e^{i(\vec{k}\cdot\vec{x}-kt)}\bar{m}_c\bar{m}_d + a^*_{(-)}(\vec{k})e^{-i(\vec{k}\cdot\vec{x}-kt)}m_cm_d).\end{aligned} \tag{44}$$

Here $k = |\vec{k}|$ and t is the background Minkowskian time. From (40) and (42) the only non-vanishing Poisson brackets between the mode coefficients are

$$\{a_{(\pm)}(\vec{k}), a^*_{(\pm)}(\vec{l})\} = -i\delta(\vec{k},\vec{l}). \tag{45}$$

In quantum theory, $\hat{a}_{(+)}(\vec{k})$ and $\hat{a}_{(-)}(\vec{k})$ are represented as annihilation operators for positive and negative helicity gravitons of wave number $\vec{k}$ and $\hat{a}^\dagger_{(+)}(\vec{k})$ and $\hat{a}^\dagger_{(-)}(\vec{k})$ are the corresponding creation operators.

4b. r-Fock states as elements of Φ^{*L}_{phys}

It is straightforward to show that the reduced phase space in the connection based description of section 2 is naturally coordinatized by the symmetric, transverse, traceless part of A_{ab} and the transverse, tracelss part of h^{ab} (recall that h^{ab} is symmetric). From (5),(6) and (7) it follows that

$$h^{abTT} = -\alpha^{abTT} \tag{46}$$

and that

$$A^{STT}_{af} = \epsilon_f^{\ cd}\frac{\partial_c \alpha^{TT}_{ad}}{2} + \gamma_0 P^{TT}_{af}. \tag{47}$$

Using (44) to express the Fourier transform of $\hat{A}^{STT}_{af}(\vec{x})$ on Fock space in terms of creation and annihilation operators, we get

$$\begin{aligned} \hat{A}^{STT}_{ab}(\vec{k}) \;&=\; \frac{\sqrt{k}m_a m_b}{2}(\hat{a}_{(+)}(\vec{k})[1-i\gamma_0] + \hat{a}^{\dagger}_{(+)}(-\vec{k})[1+i\gamma_0] \\ &+\; \frac{\sqrt{k}\bar{m}_a\bar{m}_b}{2}(\hat{a}_{(-)}(\vec{k})[-1-i\gamma_0] + \hat{a}^{\dagger}_{(-)}(-\vec{k})[-1+i\gamma_0]). \end{aligned} \tag{48}$$

We define the smeared holonomy (also called the r-holonomy) labelled by $\boldsymbol{\alpha}, \{\mathbf{q}\}$ as

$$H^{STT}_{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})} := \exp i \int d^3k X^{ab}_{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(-\vec{k}) A^{STT}_{ab}(\vec{k}) \tag{49}$$

where

$$X^{ab}_{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{k}) = e^{\frac{-k^2r^2}{2}} X^{ab}_{\boldsymbol{\alpha},\{\mathbf{q}\}}(\vec{k}). \tag{50}$$

Poincare invariance is fed into the construction of the Fock space representation through the specific choice of complex structure (i.e. the positive- negative frequency decomposition (44)). This choice is equivalent to the requirement that the Fock vacuum be a zero eigenstate of the annhilation operators. This requirement can, in turn, be encoded in terms of the smeared holonomy operators as

$$\begin{aligned} &\exp\,(i\frac{\gamma_0}{4}\int d^3x X^{ab}_{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{x}) G^{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}_{ab}(\vec{x})) \quad \exp\,(\tfrac{i}{2}\textstyle\int d^3x G^{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}_{ab}(\vec{x})\hat{h}^{ab}(\vec{x}))|0> \\ &\qquad = \hat{H}^{STT}_{\boldsymbol{\alpha},\{-\mathbf{q}\}(\mathbf{r})}|0> \end{aligned} \tag{51}$$

where $G_{ab}^{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{x})$ is defined through its Fourier transform,

$$G_{ab}^{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{k}) = kX^{+}_{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{k})(1+i\gamma_0)m_a m_b - kX^{-}_{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{k})(1-i\gamma_0)\bar{m}_a\bar{m}_b \tag{52}$$

The image of this condition in the r-Fock representation is

$$\exp\,(i\frac{\gamma_0}{4}\int d^3x X^{ab}_{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{x})G_{ab}^{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{x}))\quad \exp\,(\tfrac{i}{2}\int d^3x G_{ab}^{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{x})\hat{h}^{ab}_r(\vec{x})|0_r>\,)$$
$$= \hat{H}^{STT}_{\boldsymbol{\alpha},\{-\mathbf{q}\}}|0_r> \tag{53}$$

where

$$h^{ab}_r(\vec{k}) = e^{\frac{-k^2r^2}{2}}h^{ab}(\vec{k}) \tag{54}$$

and

$$H^{STT}_{\boldsymbol{\alpha},\{\mathbf{q}\}} = \exp i\int d^3x X^{ab}_{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{x})A^{STT}_{ab}(\vec{x}). \tag{55}$$

The r-Fock vacuum bra, $<0_r|$, can be identified with the element $\Phi_0 \in \Phi^{*L}_{phys}$ via the following equation in the dual representation (see (30)). Let $|\psi>$ be a finite linear combination of flux network states. Then

$$\Phi_0\Big(\exp\,(-i\frac{\gamma_0}{4}\int d^3x X^{ab}_{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{x})(G_{ab}^{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{x}))^*)\quad \exp\,(-\tfrac{i}{2}\int d^3x (G_{ab}^{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{x}))^*\hat{h}^{ab}_r(\vec{x})|\psi>\Big)$$
$$= \Phi_0(\hat{H}^{\dagger}_{\boldsymbol{\alpha},\{-\mathbf{q}\}}|\psi>). \tag{56}$$

Here $\hat{H}^{\dagger}_{\boldsymbol{\alpha},\{-\mathbf{q}\}}$ is defined through

$$\begin{aligned} H_{\boldsymbol{\alpha},\{\mathbf{q}\}} &= \exp i\int d^3x X^{ab}_{\boldsymbol{\alpha},\{\mathbf{q}\}}(\vec{x})A_{ab}(\vec{x}) \\ &= \prod_{k=1}^{3}\exp i\int d^3x X^{a}_{\alpha_k,\{q_k\}}(\vec{x})A^k_a(\vec{x}). \end{aligned} \tag{57}$$

Note that in (56) we have effectively replaced A^{STT}_{ab} in (49)by A^i_a. This is correct because the operator $\hat{H}_{\boldsymbol{\alpha},\{-\mathbf{q}\}}$ is defined on *physical states*. Since such states are in the kernel of the constraints and since $H_{\boldsymbol{\alpha},\{-\mathbf{q}\}}$ is a Dirac observable, the STT condition is automatically enforced on Φ^{*L}_{phys}.

As in (31) we set

$$\Phi_0 := \sum_{\boldsymbol{\alpha},\{\mathbf{q}\}} c_{0\boldsymbol{\alpha},\{\mathbf{q}\}} <\boldsymbol{\alpha},\{\mathbf{q}\}| \tag{58}$$

and solve for the coefficients $c_{0\boldsymbol{\alpha},\{\mathbf{q}\}}$ The unique (upto an overall multiplicative constant) solution is

$$c_{0\boldsymbol{\alpha},\{\mathbf{q}\}} = \exp\left(-i\frac{\gamma_0}{4}\int d^3x G_{ab}^{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{x})\right)^* X^{ab}_{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{x})). \tag{59}$$

4c. Open technical issues

In [20], it was shown that the set of states obtained by the action of the holonomy operators on the r-Fock vacuum is dense in the r-Fock space. Denote this set by D. A corresponding set of distributions, $\mathcal{D}^*$, in the (dual) loop representation was obtained by the dual action of the holonomy operators on Φ_0. The inner product between two elements of $\mathcal{D}^*$ was defined to be equal to the r-Fock inner product between the two corresponding elements of D (see (45) of [10]). This procedure is consistent provided the set of distributions in $\mathcal{D}^*$ corresponding to any (finite) linearly independent set of vectors in D, is linearly independent in $\mathcal{D}^*$. A cursory glance at this provisio indicates that its validity is very plausible but a proof, as yet, does not exist. [5]

In the case of linearised gravity, it should be straightforward to generalise the results of [20] to show that the set of states obtained by the action of the operators $\hat{H}^{STT}_{\boldsymbol{\alpha},\{\mathbf{q}\}}$ on $|0_r>$ generates a dense subspace, D_{r-Fock}, of the r-graviton Fock space. The corresponding set, Φ^{*L}_{r-Fock} can be identified by the dual action of $\hat{H}_{\boldsymbol{\alpha},\{\mathbf{q}\}}$ on Φ_0 and the inner product on Φ^{*L}_{r-Fock} can be induced from that on D_{r-Fock} by

$$(\hat{H}_{\boldsymbol{\alpha},\{\mathbf{q}\}}\Phi_0, \hat{H}_{\boldsymbol{\beta},\{\mathbf{p}\}}\Phi_0) =< 0_r|\hat{H}^{\dagger STT}_{\boldsymbol{\beta},\{\mathbf{p}\}}\hat{H}^{STT}_{\boldsymbol{\alpha},\{\mathbf{q}\}}|0_r > \tag{60}$$

Further, Φ^{*L}_{r-Fock} can be completed to a Hilbert space naturally isomorphic to the r-Fock space. Again, the procedure is consistent provided every finite linearly independent set of vectors in D_{r-Fock} defines a corresponding linearly independent set of vectors in Φ^{*L}_{r-Fock}. This remains to be shown but seems to be quite plausible.

We close with some remarks on the incorporation of the reality properties of the phase space variables in terms of adjointness properties of appropriate quantum operators.

[5] We did not realise the necessity of proving this in [10].

Modulo the open issues above, note that:

(1) the operators $\hat{H}_{\boldsymbol{\alpha},\{\mathbf{q}\}}$ and

$$\hat{M}_{\boldsymbol{\beta},\{\mathbf{p}\}(\mathbf{r})} := \exp\left(\frac{i}{2}\int d^3x G_{ab}^{\boldsymbol{\beta},\{\mathbf{p}\}(\mathbf{r})}(\vec{x})\hat{h}_{(r)}^{ab}(\vec{x})\right). \tag{61}$$

provide an (anti-) representation on Φ^{*L}_{phys} of the corresponding Poisson bracket algebra.

(2) the action on Φ_0 of the operator $\hat{M}_{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}$ is uniquely determined in terms of that of $\hat{H}_{\boldsymbol{\alpha},\{\mathbf{q}\}}$ from (56). This, in conjunction with (1), implies that the action of the operators $\hat{H}_{\boldsymbol{\alpha},\{\mathbf{q}\}}$ and $\hat{M}_{\boldsymbol{\beta},\{\mathbf{p}\}(\mathbf{r})}$ on Φ^{*L}_{r-Fock} is naturally isomorphic to the action of the corresponding operators on D_{r-Fock} in the (dual) r-Fock representation.

(3)the r-Fock inner product correctly enforces the adjointness properties of these operators in the r-Fock representation.

From (1)- (3) above, it is reasonable to expect that the inner product (60) incorporates the appropriate reality conditions. However, an explicit proof of this is still lacking and is expected to be a bit involved for the following reason. The function $G_{ab}^{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{x})$ is complex. As a result, $\hat{M}_{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}$ is neither unitary nor hermitian and consequently it is expected that the algebra of operators generated by $\hat{M}_{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}$ and $\hat{H}_{\boldsymbol{\alpha},\{\mathbf{q}\}}$ is not closed under the adjoint operation, thus complicating the required proof.

5. The case of complex γ_0

Our considerations till now have been based on the real $SU(2)$ formulation of gravity. Remarkably, much of our analysis can also be applied to the formulation of section 2 with an arbitrary *complex* Barbero- Immirzi parameter, γ_0, including the case of $\gamma_0 = -i$ which corresponds to the choice of self dual variables [14, 11].

We adopt the viewpoint that the kinematic $U(1)^3$ based Hilbert space, $\mathcal{H}_{kin}$, is simply an auxilliary structure whose only role is to furnish a (dual) representation of the algebra (*not* the * algebra) generated by $H_{\boldsymbol{\alpha},\{\mathbf{q}\}}$ and h^{ab}. This representation is to be used to find the kernel of the quantum constraints and the physical inner product

is to be chosen in such a way as to enforce the * relations on Dirac observables as adjointness relations on the corresponding operators.

To this end, the analysis of sections 2 and 3 holds with a complex γ_0. Note that the dual representation is defined by (30) with the adjoint operation taken *with respect to the kinematic Hilbert space inner product*. For γ_0 complex, this 'kinematic' adjoint operation does *not* enforce the * relations obtained from the 'reality conditions' [11]. The reality conditions on the linearised variables are

$$(h^{ab})^* = h^{ab} \qquad (\frac{A_{(ab)} - \Gamma_{(ab)}}{\gamma_0})^* = (\frac{A_{(ab)} - \Gamma_{(ab)}}{\gamma_0}) \tag{62}$$

and are to be incorporated in the quantum theory by the physical inner product, not necessarily the kinematic one. In fact, with respect to the kinematic adjoint operation, $\hat{h}^{ab}$ is not self adjoint. Instead in contrast to (26) we have that

$$\hat{h}^{\dagger ab}(\vec{x})|\boldsymbol{\alpha}, \{\mathbf{q}\}> = \gamma_0^* X^{(ab)}_{\boldsymbol{\alpha},\{\mathbf{q}\}}(\vec{x})|\boldsymbol{\alpha}, \{\mathbf{q}\}> \tag{63}$$

The contents of section 4b upto and including (55) are valid even for complex γ_0. In particular, the Poincare invariance of the vacuum is still encoded in (51). Equation (53) too, is unchanged but (56) in the dual representation must be defined through (30). Since $X^{(ab)}_{\boldsymbol{\alpha},\{\mathbf{q}\}}(\vec{x})$ is real and since (with the kinematic adjoint) $\hat{h}^{\dagger ab} \neq \hat{h}^{ab}$ when γ_0 is complex, (56) is replaced by

$$\Phi_0\Big(\exp{(-i\frac{\gamma_0^*}{4}\int d^3x X^{ab}_{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{x})(G_{ab}^{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{x}))^*)} \quad \exp{(-\tfrac{i}{2}\int d^3x (G_{ab}^{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{x}))^* \hat{h}_r^{\dagger ab}(\vec{x})|\psi>} \Big)$$
$$= \Phi_0(\hat{H}^{\dagger}_{\boldsymbol{\alpha},\{-\mathbf{q}\}}|\psi>). \tag{64}$$

This equation admits the unique (upto an overall multiplicative constant) solution

$$c_{0\boldsymbol{\alpha},\{\mathbf{q}\}} = \exp{(-i\frac{\gamma_0^*}{4}\int d^3x (G_{ab}^{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{x}))^* X^{ab}_{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}(\vec{x}))}. \tag{65}$$

When $\gamma_0 \neq \pm i$, we again expect the steps of section 4c to go through with the inner product on Φ^{*L}_{Fock} specified through

$$(\hat{H}_{\boldsymbol{\alpha},\{\mathbf{q}\}}\Phi_0, \hat{H}_{\boldsymbol{\beta},\{\mathbf{p}\}}\Phi_0) =< 0_r|\hat{H}^{\dagger STT}_{\boldsymbol{\beta},\{\mathbf{p}\}}\hat{H}^{STT}_{\boldsymbol{\alpha},\{\mathbf{q}\}}|0_r> \tag{66}$$

where $\hat{H}^{\dagger STT}_{\boldsymbol{\beta},\{\mathbf{p}\}}$ is the adjoint with respect to the r- Fock inner product. The latter correctly incorporates the reality conditions given by (62). In particular, since γ_0 is

complex, $\hat{H}^{STT}_{\boldsymbol{\beta},\{\mathbf{p}\}}$ is *not* a unitary operator. Note that the comments in section 4c regarding the incorporation of reality conditions in terms of adjointness conditions also apply to the inner product (66).

When $\gamma_0 = i$ or $-i$, (48) implies that $\hat{A}^{STT}_{ab}(\vec{k})$ lacks either the positive helicity creation operator or the negative helicity creation operator. Hence $\hat{H}^{STT}_{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}$ cannot generate the positive helicity (respectively, negative helicity) graviton sector from the vacuum. Instead, operators involving the linearised metric would have to be used to generate the Hilbert space from the vacuum. Although we have not attempted the relevant analysis, we do expect that the methods of [11] can be recast in the language of this paper to successfully do so.

6. Discussion

In this work we have shown how the r-Fock representation for linearised gravity can be constructed, starting from the 'loop' representation on the kinematic Hilbert space, $\mathcal{H}_{kin}$. The role of the representation on $\mathcal{H}_{kin}$ in this construction is that it provides the structure to define, with mathematical precision, the dual (anti-)representation on an appropriate space of distributions. In particular, the role of the kinematic Hilbert space inner product is to define the kinematic adjoint operation which is, in turn, used to define the dual representation through (30).

One of the features of this work is that it highlights the importance of the dual representation on the space of distributions. Physical states (as opposed to kinematic ones) lie in this space. The condition (53) which is satisfied by the r- Fock vacuum in the r-Fock representation not only makes sense (in the form of (56)), but also admits an essentially unique solution, Φ_0, in the dual representation. Modulo the comments in sections 4c and 5, the rest of (a dense subspace of) the r- Fock space is then generated from Φ_0, once again, via the dual representation of appropriately chosen Dirac observables.

Another feature of this work is that the inner product on physical states, namely the r- Fock inner product, is very different from the kinematic inner product. Indeed,

the physical states are not kinematically normalizable. The results of section 5 further de-emphasize the *physical* significance of the kinematic inner product and seem to strengthen the old viewpoint in the loop quantum gravity approach wherein the physical inner product is to be determined by the reality conditions. Note, however, that the kinematic structures continue to play a key *mathematical* role in defining the dual representation even when γ_0 is complex. Thus, even though the rigorous mathematical structures of [6, 15, 7, 4] are defined only for compact gauge groups, we were able to use such structures profitably, even for the self dual description of linearised gravity.

We now turn to a brief discussion of the physical indistinguishability of the r-Fock and the Fock representations. In the $U(1)$ context we noted in [10] that there were two possible viewpoints with regard to this issue. One viewpoint is that only *algebraic* properties of functions on phase space are measurable. This viewpoint applied to linearised gravity would imply that there is no way of asserting whether the pair $(H^{STT}_{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}, h^{abSTT})$ is being measured in the Fock representation or the pair $(H^{STT}_{\boldsymbol{\alpha},\{\mathbf{q}\}}, h^{abSTT}_{(r)})$ is being measured in the r-Fock representation. Thus, with this viewpoint, the physics of the r-Fock representation is exactly (not approximately) *identical* to that of the Fock representation.

The other viewpoint is valid in the case that there is some property other than purely algebraic properties of the pair $(H^{STT}_{\boldsymbol{\alpha},\{\mathbf{q}\}(\mathbf{r})}, h^{abSTT})$ by virtue of which the measuring apparatus measures them rather than the pair, $(H^{STT}_{\boldsymbol{\alpha},\{\mathbf{q}\}}, h^{abSTT}_{(r)})$. In such a case, the r-Fock representation is physically indistinguishable from the Fock representation only for finite accuracy measurements at distance scales much larger than r [10]. Linearised gravity is a truncation of full general relativity. In the latter, the primary object which is measured is the full metric. The notion of smearing does not extend to an arbitrary metric in any natural way (note that the smearing we use is heavily dependent on the background flat metric). Thus, for a reason external to the narrow confines of linearised gravity, we expect that the physical apparatus measures the combination $\delta^{ab} + h^{ab}$ from which h^{ab} can be estimated. Hence the object h^{ab} rather than $h^{ab}_{(r)}$ is preferred and the second viewpoint mentioned above seems to

be the valid one. We have explored consequences of this viewpoint for violation of Poincare invariance at scales smaller than r and will report our results elsewhere [23].

As mentioned in the introduction, the deeper question of how (if at all!) the $U(1)^3$ loop representation arises from loop quantum gravity is as yet unsolved. A small preliminary step in this direction would be to investigate if the linearised constraints can be solved via an 'averaging' procedure [24] similar to that used in loop quantum gravity [4], rather than by using the magnetic field operator. This would bring the $U(1)^3$ approach structurally even closer to the loop quantum gravity approach.

This work represents the culmination of our efforts, initiated in [20] and continued in [10], to understand the older results of [11] in the mathematically precise language currently used in the field. We hope that this work may aid current efforts to construct semiclassical states in loop quantum gravity [22, 25] and suggest that it may be a profitable venture to revisit the older efforts of Iwasaki and Rovelli [26] in the light of subsequent developments in the field.

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